Vacuum-induced jitter in spatial solitons

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Abstract: We perform a calculation to determine how quantum mechanical fluctuations influence the propagation of a spatial soliton through a nonlinear material. To do so, we derive equations of motion for the linearized operators describing the deviation of the soliton position and transverse momentum from those of a corresponding classical solution to the nonlinear wave equation, and from these equations we determine the quantum uncertainty in the soliton position and transverse momentum. We find that under realistic laboratory conditions the quantum uncertainty in position is several orders of magnitude smaller the classical width of the soliton. This result suggests that the reliability of photonic devices based on spatial solitons is not compromised by quantum fluctuations.

OCIS codes: (270.5530) Pulse propagation and solitons; (190.0190) Nonlinear optics

References and links

1. A pedagogical discussion of self-action effects including self-trapping and optical solitons is presented in Chapter 6 of R. W. Boyd, *Nonlinear Optics* (Academic, San Diego, 1992). See also E. M. Nagasako, R. W. Boyd, in *Amazing Light, A volume dedicated to Charles Hard Townes on his 80th Birthday*, edited by R. Y. Chiao (Springer, New York, 1996).
2. G. A. Askar'yan, Sov. Phys. JETP **15**, 1088 (1962).
3. R. Y. Chiao, E. Garmire, and C. H. Townes, Phys. Rev. Lett. **13**, 479 (1964).
4. V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP **34**, 62 (1972).
5. E. L. Dawes and J. H. Marburger, Phys. Rev. **179**, 862 (1969); See also J. H. Marburger, Prog. Quantum Electron. **4**, 35 (1975).
6. Y. Silberberg, Opt. Lett. **15**, 1282 (1990).
7. J. S. Aitchison, Y. Silberberg, A. M. Weiner, D. E. Leaird, M. K. Oliver, J. L. Jackel, E. M. Vogel, and P. W. E. Smith, J. Opt. Soc. Am. B **8**, 1290 (1991).
8. M. Shalaby and A. Barthelemy, Opt. Comm. **94**, 341 (1992).
9. A. Villeneuve, J. S. Aitchison, J. U. Kang, P. G. Wigley, and G. I. Stegeman, Opt. Lett. **19**, 761 (1994).
10. B. Luther-Davies and X. Yang, Opt. Lett. **17**, 1755 (1992).
11. S. Blair, K. Wagner, and R. McLeod, Opt. Lett. **19**, 1943 (1994).
12. E. M. Nagasako, R. W. Boyd, and G. S. Agarwal, Phys. Rev. A **55**, 1412 (1997).
13. J. D. Gordon and H. A. Haus, Opt. Lett. **11**, 665 (1986).
14. P. D. Drummond R. M. Shelby, S. R. Friberg, and Y. Yamomoto, Nature, **365**, 307 (1993).
15. H. A. Haus and M. N. Islam, IEEE J. Quantum Electron. **QE-21**, 1172 (1985).
16. H. A. Haus and Y. Lai, J. Opt. Soc. Am. B **7**, 386 (1990).
17. Although c has the fixed value $-1/2$ using the present conventions, we retain c in our formulas for more ready comparisons of our results with those obtained using different normalization conventions.

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1. Introduction

Spatial solitons are beams of light that propagate with a constant transverse dimensions as the result of an exact balance between diffraction and self focusing effects [1]. The possibility of such an occurrence was recognized in some of the very early studies of nonlinear optical phenomena [2,3], and the relation between spatial solitons and temporal solitons was elucidated by Zakharov and Shabat [4]. Although the balance between diffraction and self focusing can occur in either one or two transverse dimensions, the equilibrium condition is unstable in the two-dimensional case for a material with a purely third order nonlinear susceptibility [5,6]. Spatial solitons, however, are predicted to be stable and have been observed experimentally in one transverse dimension [7-9], as occurs for instance in the case of propagation through a planar waveguide.

Numerous technological applications based on the properties of spatial solitons have been proposed, including optical interconnects [10], optical logic gates [11], and other optical switching devices. The operational integrity of any such device requires that the spatial jitter in the position of the spatial soliton be held to a minimum. In the present paper, we describe a theoretical investigation of the influence of quantum noise on the propagation of spatial solitons, and in particular derive expressions that allow us to determine the uncertainty in the position of a spatial soliton after propagating through a nonlinear optical medium. As described below, we find that quantum fluctuations can lead to a small but measurable uncertainty in the soliton position. We earlier studied the influence of quantum fluctuations on the initiation of the filamentation process [12].

There is a considerable body of prior work involving the influence of quantum noise on the properties of temporal solitons. Much of this work was motivated by issues associated with the reliability of optical fiber telecommunications systems. For instance, Gordon and Haus [13] have considered the influence of noise associated with the amplification of solitons signal trains in a long-distance telecommunications system. The influence of quantum fluctuations on propagation of solitons through a passive optical fiber system has also been considered, for instance by Drummond and co-workers [14] and by Haus and co-workers [15,16]. The calculation presented in the present paper follows much of the same methodology as that used by Haus and Lai [16] in their treatment of temporal solitons. Our work differs from theirs in that we use the formalism to treat spatial rather than temporal solitons. In addition, we are particularly interested in determining the uncertainty in the transverse soliton position, whereas Haus and Lai used their formalism to treat quadrature squeezing in the transmitted soliton. In addition, we establish some mathematical relations such as the definition of the adjoint operation in the context of the present problem (Appendix A) and we establish the sense in which soliton propagation can be described by the same equations as the quantum mechanical treatment of a free particle (Appendix B).

2. Theoretical Formulation

Our theoretical development starts with the nonlinear Schrodinger equation with variation in one transverse spatial dimension

$$2ik\frac{\partial}{\partial z}A + \frac{\partial^2}{\partial x^2}A + 2k^2\frac{\bar{n}_2}{n_0}|A|^2 A = 0\,, \tag{1}$$

where the amplitude A is defined according to the convention

$$E(r,t) = \frac{1}{2}\hat{x}\left[A(x,z)\exp\left(i\beta_0 z - i\omega_0 t\right) + c.c.\right]\,. \tag{2}$$

We make the change of variables

$$X \equiv \sqrt{2}\,kx \qquad Z \equiv kz \qquad c \equiv -1/2$$
$$\Phi^{(+)} \equiv \frac{A}{\sqrt{n_0/\bar{n}_2}} \qquad \Phi^{(-)} \equiv \left(\Phi^{(+)}\right)^* \equiv \frac{A^*}{\sqrt{n_0/\bar{n}_2}}$$

where $\bar{n}_2 = (n_0 c/4\pi)n_2$ where n_2 is the usual nonlinear refractive index defined such that $\Delta n = n_2 I$. We then find that the nonlinear Schrodinger equation (for the positive frequency part of the field) takes the standard [16] normalized form [17]

$$i\frac{\partial}{\partial Z}\Phi^{(+)} + \frac{\partial^2}{\partial X^2}\Phi^{(+)} - 2c\left|\Phi^{(+)}\right|^2\Phi^{(+)} = 0\,. \tag{3}$$

The fundamental soliton solution to this equation is given by

$$\begin{aligned}\Phi_0^{(+)}(X,Z) &= \frac{N_0|c|^{1/2}}{2}\exp\left[i\frac{N_0^2|c|^2}{4}Z - ip_0^2Z + ip_0(X-X_0) + i\theta_0\right] \\ &\quad \operatorname{sech}\left[\frac{N_0|c|}{2}(X - X_0 - 2p_0Z)\right]\,,\end{aligned} \tag{4}$$

where the quantity

$$N_0 = \int \left|\Phi^{(+)}(X,Z)\right|^2 dX$$

represents the soliton intensity profile integrated over the transverse spatial dimension X, θ_0 is the soliton phase, p_0 is the transverse soliton momentum, and X_0 is the position of the soliton center.

Let us next consider how the spatial evolution of this soliton solution is influenced by small perturbations in the solitons parameters. To do so, we represent the field amplitude as the sum of the fundamental soliton solution and a small perturbation $\hat{\Psi}^{(+)}$ which we treat quantum mechanically:

$$\hat{\Phi}^{(+)}(X,Z) = \Phi^{(+)}(X,Z) + \hat{\Psi}^{(+)}(X,Z)\,. \tag{5}$$

We also introduce the negative frequency part of the field operators

$$\hat{\Psi}^{(-)}(X,Z) = \hat{\Psi}^{(+)}(X,Z)^\dagger\,, \qquad \hat{\Phi}^{(-)}(X,Z) = \hat{\Phi}^{(+)}(X,Z)^\dagger\,,$$

etc. The perturbation is assumed to obey the quantum mechanical commutation relation

$$\left[\hat{\Psi}^{(+)}(X',Z), \hat{\Psi}^{(-)}(X,Z)\right] = \mathcal{S}\delta(X - X') \qquad \mathcal{S} = \frac{\hbar c k_0^2 n_2}{2\,\Delta t\, L_y}\,. \tag{6}$$

where Δt is the response time of the nonlinear response and L_y is the thickness of the slab waveguide that confines the radiation in the y direction. The form of the coefficient $\mathcal{S}$ is obtained by requiring that the total electric field obey the standard field commutation relation and then reducing the problem to one transverse dimension and one frequency component, under the assumptions that the spectrum is essentially uniform over the frequency interval $1/\Delta t$ and that the field amplitude is essentially constant over the thickness L_y of the waveguide. By substituting expression (5) into the nonlinear Schrodinger equation (1) and linearizing in the perturbation, the evolution equation for the perturbation

$$i\frac{\partial}{\partial Z}\hat{\Psi}^{(+)} + \frac{\partial^2}{\partial X^2}\hat{\Psi}^{(+)} + 4|c|\left|\Phi_0^{(+)}\right|^2\hat{\Psi}^{(+)} + 2|c|\left(\Phi_0^{(+)}\right)^2\hat{\Psi}^{(-)} = 0 \tag{7}$$

is obtained. It has been shown previously [16] that to a good approximation the perturbation $\hat{\Psi}^{(+)}$ can be represented as the sum of contributions resulting from fluctuations in the four fundamental soliton degrees of freedom:

$$\hat{\Psi}^{(+)}(X,Z) = \frac{\partial\Phi_0^{(+)}}{\partial N_0}\Delta\hat{N}_0^{(+)} + \frac{\partial\Phi_0^{(+)}}{\partial\theta_0}\Delta\hat{\theta}_0^{(+)} + \frac{\partial\Phi_0^{(+)}}{\partial p_0}\Delta\hat{p}_0^{(+)} + \frac{\partial\Phi_0^{(+)}}{\partial X_0}\Delta\hat{X}_0^{(+)} . \tag{8}$$

In writing this equation in the form shown, we have ignored the continuum contribution to the perturbation. The derivatives which form the coefficients of the fluctuation operators in this expression are also solutions of Eq. (7). These solutions can be denoted by

$$\Psi_j^{(+)}(X,Z) = \frac{\partial\Phi_0^{(+)}(X,Z)}{\partial j_0}\,, \tag{9}$$

where $j = N, \theta, p$ and X and are given explicitly by

$$\begin{aligned}
\Psi_N^{(+)}(X,Z) &= \left[\frac{1}{N_0} + i\frac{N_0|c|^2}{2}Z - \frac{c}{2}X\,\tanh\left(\frac{N_0|c|^2}{2}X\right)\right]\Phi_0^{(+)}(X,Z)\,,\\
\Psi_\theta^{(+)}(X,Z) &= i\Phi_0^{(+)}(X,Z)\,,\\
\Psi_p^{(+)}(X,Z) &= \left[iX + N_0|c|Z\,\tanh\left(\frac{N_0|c|}{2}X\right)\right]\Phi_0^{(+)}(X,Z)\,,\\
\Psi_X^{(+)}(X,Z) &= \left[\frac{N_0|c|}{2}\,\tanh\left(\frac{N_0|c|}{2}X\right)\right]\Phi_0^{(+)}(X,Z)\,.
\end{aligned} \tag{10}$$

Using the derivative solutions $\Psi_j^{(+)}(X,Z)$ as well as their adjoints $\underline{\Psi}_j^{(+)}(X,Z)$ (see Appendix A for a description of the adjoint operation in the present context), we can now invert the expansion of the perturbation operator $\hat{\Psi}^{(+)}$ to give the individual fluctuation operators as follows :

$$\begin{aligned}
\Delta\hat{N}_0 &= -\int\left[\underline{\Psi}_\theta^{(-)}(X,0)\hat{\Psi}^{(+)}(X,0) + \underline{\Psi}_\theta^{(+)}(X,0)\hat{\Psi}^{(-)}(X,0)\right]dX\,,\\
\Delta\hat{\theta}_0 &= \int\left[\underline{\Psi}_N^{(-)}(X,0)\hat{\Psi}^{(+)}(X,0) + \underline{\Psi}_N^{(+)}(X,0)\hat{\Psi}^{(-)}(X,0)\right]dX\,,\\
\Delta\hat{p}_0 &= \frac{1}{N_0}\int\left[\underline{\Psi}_X^{(-)}(X,0)\hat{\Psi}^{(+)}(X,0) + \underline{\Psi}_X^{(+)}(X,0)\hat{\Psi}^{(-)}(X,0)\right]dX\,,\\
\Delta\hat{X}_0 &= -\frac{1}{N_0}\int\left[\underline{\Psi}_p^{(-)}(X,0)\hat{\Psi}^{(+)}(X,0) + \underline{\Psi}_p^{(+)}(X,0)\hat{\Psi}^{(-)}(X,0)\right]dX\,.
\end{aligned} \tag{11}$$

Here

$$\Delta\hat{N}_0 = \Delta\hat{N}_0^{(+)} + \Delta\hat{N}_0^{(-)} = \Delta\hat{N}_0^{(+)} + H.c.$$

represents the complete (i.e. Hermitian) fluctuation operator, and $\Delta\hat{N}_0^{(+)}$ its positive frequency part, and so on. Eqs. (11) can be verified by direct calculation making use of Eqs. (8) and (10). Note that these are the fluctuation operators relevant to the entrance plane to the material. Using these expressions, the following commutation relationships involving the fluctuation operators can be calculated:

$$\begin{aligned}
\left[\Delta\hat{N}_0, \Delta\hat{\theta}_0\right] &= i\mathcal{S}\\
\left[\Delta\hat{X}_0, N_0\Delta\hat{p}_0\right] &= i\mathcal{S}\,.
\end{aligned} \tag{12}$$

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These commutation relations are consistent with the view that the pair $\Delta\hat{N}_0$ and $\Delta\hat{\theta}_0$ and the pair $\Delta\hat{X}_0$ and $N_0\Delta\hat{p}_0$ form conjugate operator pairs, and lead to uncertainty relations between the conjugate quantities. We shall see below (Eq. (22)) that the parameter N_0 is typically smaller than unity, with the consequence that uncertainty relation expressed in terms of position $\Delta\hat{X}_0$ and momentum $\Delta\hat{p}_0$ (without the factor of N_0) predicts a quantum mechanical uncertainty increased over the single particle case by a factor of $1/N_0$.

The fluctuation operators at any position Z within the interaction region can be determined similarly. For simplicity, we consider only the case of a soliton propagating along the Z axis, that is, we set p_0 and X_0 equal to zero. We then obtain the results

$$\begin{aligned}
\Delta\hat{N}(Z) &= -\int\left[\underline{\Psi}_\theta^{(-)}(X,0)\exp\left(-i\frac{N_0^2|c|^2Z}{4}\right)\hat{\Psi}^{(+)}(X,Z)\right. \\
&\qquad \left.+\underline{\Psi}_\theta^{(+)}(X,0)\exp\left(i\frac{N_0^2|c|^2Z}{4}\right)\hat{\Psi}^{(-)}(X,Z)\right]dX\,, \\
\Delta\hat{\theta}(Z) &= \int\left[\underline{\Psi}_N^{(-)}(X,0)\exp\left(-i\frac{N_0^2|c|^2Z}{4}\right)\hat{\Psi}^{(+)}(X,Z)\right. \\
&\qquad \left.+\underline{\Psi}_N^{(+)}(X,0)\exp\left(i\frac{N_0^2|c|^2Z}{4}\right)\hat{\Psi}^{(-)}(X,Z)\right]dX\,, \\
\Delta\hat{p}(Z) &= \frac{1}{N_0}\int\left[\underline{\Psi}_X^{(-)}(X,0)\right.
\end{aligned}$$

$$\exp\left(-i\tfrac{N_0^2|c|^2Z}{4}\right)\hat{\Psi}^{(+)}(X,Z)$$
$$\left.+\underline{\Psi}_X^{(+)}(X,0)\exp\left(i\tfrac{N_0^2|c|^2Z}{4}\right)\hat{\Psi}^{(-)}(X,Z)\right]dX\,,(13)$$
$$\Delta\hat{X}(Z) = -\tfrac{1}{N_0}\int\left[\underline{\Psi}_p^{(-)}(X,0)\exp\left(-i\tfrac{N_0^2|c|^2Z}{4}\right)\hat{\Psi}^{(+)}(X,Z)\right.$$
$$\left.+\underline{\Psi}_p^{(+)}(X,0)\exp\left(i\tfrac{N_0^2|c|^2Z}{4}\right)\hat{\Psi}^{(-)}(X,Z)\right]dX\,.$$

We mentioned above in connection with Eqs. (12) that $\Delta\hat{X}_0$ and $N_0\Delta\hat{p}_0$ obey quantum mechanical commutation relations. Moreover, we show in Appendix B that the soliton position fluctuation operator $\Delta\hat{X}$ and momentum fluctuation operator $\Delta\hat{p}$ obey the equations

$$\begin{aligned}
\frac{d}{dZ}\Delta\hat{p} &= 0 \\
\frac{d}{dZ}\Delta\hat{X} &= 2\,\Delta\hat{p}\,,
\end{aligned} \tag{14}$$

which are the quantum mechanical equations of motion for a free particle with the formal substitution $Z \to t$. The factor of 2 in the second of Eqs. (14) results as a consequence of the particular conventions used in this calculation. These equations can be integrated to express the output fluctuations in terms of the input fluctuations as follows

$$\begin{aligned}
\Delta\hat{p}(Z) &= \Delta\hat{p}_0 \\
\Delta\hat{X}(Z) &= \Delta\hat{X}_0 + 2\,\Delta\hat{p}_0\,Z\,.
\end{aligned} \tag{15}$$

We can now use these results to determine the quantum mechanical uncertainty in the position of the soliton and to determine how this uncertainty changes with propagation distance Z. As before, we consider the case of a soliton propagating along the Z axis, that is, we set p_0 and X_0 equal to zero. Then through use of Eqs. (11) we find that the initial uncertainties in the soliton position and momentum, that is, the uncertainties at the entrance plane ($Z = 0$) are given by

$$\left\langle \Delta \hat{X}_0^2 \right\rangle = \frac{\pi^2 \mathcal{S}}{3 N_0^3 |c|^2} \tag{16}$$

and

$$\left\langle \Delta \hat{p}_0^2 \right\rangle = \frac{N_0 |c|^2 \mathcal{S}}{12} \, . \tag{17}$$

We can then use the result of the second of the Eqs. (15), to find that the uncertainty at some arbitrary point Z in the material is given by

$$\left\langle \Delta X^2(Z) \right\rangle = \frac{\pi^2 \mathcal{S}}{3 N_0^3 |c|^2} + \frac{N_0 |c|^2 \mathcal{S}}{3} Z^2 \, . \tag{18}$$

We next express these results in terms of physical, i.e. non normalized units. To do so, we first note that the fundamental soliton solution to the non-normalized equation (1) for the propagation of a spatial soliton of width w along the z axis is

$$A(z) = A_{peak} \ \mathrm{sech}\left(\frac{x}{w}\right) \exp\left(i \frac{z}{2kw^2}\right) \tag{19}$$

where

$$A_{peak} = \frac{1}{kw} \sqrt{\frac{n_0}{2\bar{n}_2}} \, . \tag{20}$$

Note that the maximum change in refractive index

$$\Delta n = \frac{1}{2} \bar{n}_2 A_{peak}^2$$

is given by

$$\Delta n = \frac{n_0}{2k^2 w^2} \, . \tag{21}$$

By comparison of Eq. (19) with Eq. (4), with p_0, X_0, and θ_0 set equal to zero, we find that the parameter N_0 of the normalized solution is given by

$$N_0 = \frac{2}{kw} \, . \tag{22}$$

Through use of Eqs. (18) through (22), and the variable change used in writing Eq. (3), we finally find that the mean squared position uncertainty in physical units can be expressed as

$$\left\langle \Delta x^2(z) \right\rangle = \frac{\pi^2 \mathcal{S} (kw) w^2}{12} + \frac{\mathcal{S}}{12(kw)} z^2 \, . \tag{23}$$

3. Discussion and Conclusions

Eq. (23) presents the primary result of the present calculation. We recall that this calculation is based on a linearization assumption, and thus strictly speaking its limits of validity require that the second term on the right hand side of this equation be much smaller than the first. Thus Eq. (23) is accurate for propagation distances that satisfy

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the inequality $z << \pi k w^2/2$. For any value of z that satisfies this condition, including $z = 0$, the uncertainty in position is dominated by the first term. Under these conditions, the fractional uncertainty in soliton position will be given approximately as

$$\frac{(\Delta x)_{rms}}{w} \approx \sqrt{\frac{\pi^2}{12} \mathcal{S}\, kw}\,.$$

We can evaluate this expression under typical laboratory conditions. We asssume that that $k_0 = 1.3 \times 10^5 \mathrm{cm}^{-1}$, corresponding to a wavelength of 0.5 μm, and that w and L_y are both of the order of 50 μm. Material parameters enter into the calculation as the ratio $n_2/\Delta t$. This ratio tends to be nearly constant for most nonlinear optical materials, because materials that display a large nonlinear respone tend to be slow. One of the largest values of this ratio occurs for the conjugated polymers, for which n_2 can be as large as 3 $\times 10^{-12} \mathrm{cm}^2/\mathrm{W}$ and for which the response time Δt is believed to be as short as 10 fs. We then find that $(\Delta x)_{rms}/w \approx 10^{-3}$, which is at best barely measurable in the laboratory. Of course, the predicted fractional uncertainty in soliton position would be larger if calculated for a material with a larger value of $n_2/\Delta t$. Note also that Eq. (23), if extrapolated to values of z outside of its demonstrated limits of validity, predicts a linear increase in the fractional uncertainly in soliton position with increasing propagation distance z. It is not clear at present what the accurately predicted uncertainty would be under these conditions.

In summary, we have presented an analysis of the influence of quantum mechanical zero-point fluctuations on the propagation of spatial solitons, and have found that these fluctuations can lead to an uncertainty in the transverse position of the soliton. For most realistic conditions with currently available materials, this predicted fractional uncertainty is at best one part in 10^3 and consequently is barely experimentally observable. However, the fractional uncertainty could be considerabley larger through use of materials with a larger nonlinear respponse or under conditions that lie outside of the validity of the present theory. These larger fractional encertainties in soliton position could have considerable importance for the construction of optical switching devices that rely on the properties of spatial solitons.

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Appendix A

In this appendix, we present a discussion of the concept of the adjoint to the classical solution of the linearized propagation equation (7). First, we note that any two classical solutions $\Psi_1^{(+)}(X,Z)$ and $\Psi_2^{(+)}(X,Z)$ to Eq. (7) must obey the equations

$$\begin{aligned} i\frac{\partial \Psi_1^{(+)}}{\partial Z} &= -L\Psi_1^{(+)} - 2c\left(\Phi_0^{(+)}\right)^2 \Psi_1^{(-)} \\ i\frac{\partial \Psi_2^{(+)}}{\partial Z} &= -L\Psi_2^{(+)} - 2c\left(\Phi_0^{(+)}\right)^2 \Psi_2^{(-)} \end{aligned} \qquad (24)$$

where the operation L is defined as

$$L \equiv \frac{\partial^2}{\partial X^2} + 4c\left|\Phi_0^{(+)}\right|^2 .$$

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By combining the first equation (24) with the complex conjugate of the second, the following relation can be deduced by direct computation:

$$i\int \frac{\partial\left(\Psi_1^{(+)}\Psi_2^{(-)}\right)}{\partial Z}\,dX = \int\left[-\Psi_2^{(-)}L\Psi_1^{(+)} + \Psi_1^{(+)}L\Psi_2^{(-)} - 2c\left(\Phi_0^{(+)}\right)^2\Psi_1^{(-)}\Psi_2^{(-)} + 2c^*\left(\Phi_0^{(-)}\right)^2\Psi_2^{(+)}\Psi_1^{(+)}\right]dX\,. \tag{25}$$

We can simplify this equation to give

$$i\frac{\partial}{\partial Z}\int\left(\Psi_1^{(+)}\Psi_2^{(-)}\right)dX = \int\left[-2c\left(\Phi_0^{(+)}\right)^2\Psi_1^{(-)}\Psi_2^{(-)} + 2c^*\left(\Phi_0^{(-)}\right)^2\Psi_2^{(+)}\Psi_1^{(+)}\right]dX\,. \tag{26}$$

In this form, it is clear that the addition of this equation with its complex conjugate gives zero, that is,

$$\frac{\partial}{\partial Z}\left[\int \Psi_1^{(+)}\left(i\Psi_2^{(-)}\right)dX + c.c.\right] = 0\,. \tag{27}$$

If we now define the adjoint of $\Psi_i^{(+)}$ by the relation

$$\underline{\Psi}_i^{(+)}(X,Z) = i\Psi_i^{(+)}(X,Z)\,, \tag{28}$$

we find that Eq. (27) can be rewritten as

$$\frac{\partial}{\partial Z}\left[\int\left(\Psi_2^{(+)}(X,Z)\underline{\Psi}_2^{(-)}(X,Z)\right)dX + c.c.\right] = 0\,. \tag{29}$$

This result shows that in the context of the present problem $\underline{\Psi}_i^{(+)}$ may be considered to be the adjoint of $\Psi_i^{(+)}$ because it expresses the fact that the inner product of two quantities remains invariant upon propagation. This of course is not the usual definition of the adjoint operation. The adjoint is usually defined within the context of an eigenvalue problem. In such a circumstance, the functions $\Psi_1^{(+)}(X,Z)$ and $\Psi_2^{(+)}(X,Z)$ would have the form

$$\Psi_i^{(+)}(X,Z) = f_i(X)\exp\left(-iE_iZ\right) \tag{30}$$

where $f_i(X)$ is the eigenfunction and E_i is the eigenvalue. If this form is substituted into Eq. (29) we find that the equation is satisfied for $E_1 \neq E_2$ if

$$\int f_1(X)f_2^*(X)\,dX = 0\,, \tag{31}$$

which is the usual definition of the adjoint operation.

Appendix B

In this appendix we prove the basic relations (14) which show that the propagation of a spatial soliton can be described in terms of the quantum mechanical equations of motion of a free particle. Through use of the third of Eqs. (13) and the last of the definitions (10) for $\Psi_X^{(+)}(X,Z)$, we find that $\Delta\hat{p}(Z)$ can be written as

$$\Delta\hat{p}(Z) = \frac{1}{N_0}\int \underline{\Psi}_X^{(+)}(X,Z)^*\,\hat{\Psi}^{(+)}(X,Z)\,dX \tag{32}$$

We note that the integrals appearing in this equation are of the form of those appearing in Eq. (29) and therefore that

$$\frac{d}{dZ}\Delta\hat{p}(Z) = 0\,. \tag{33}$$

We next consider the evolution of $\Delta\hat{X}(Z)$ as defined by last of Eqs. (13). Through use of Eqs. (10) and (28) we find that

$$\underline{\Psi}_p^{(+)}(X,Z)^* \exp\left(-i\frac{N_0^2|c|^2 Z}{4}\right) = \underline{\Psi}_p^{(+)}(X,Z)^* - 2Z\underline{\Psi}_X^{(+)}(X,Z)^* \tag{34}$$

and thus that $\Delta\hat{X}(Z)$ can be written as

$$\begin{aligned}\Delta\hat{X}(Z) \quad &= \quad -\frac{1}{N_0}\int \underline{\Psi}_p^{(+)}(X,Z)^*\,\hat{\Psi}^{(+)}(X,Z)\,dX + c.c. \\ &\quad +2Z\frac{1}{N_0}\int \underline{\Psi}_X^{(+)}(X,Z)^*\,\hat{\Psi}^{(+)}(X,Z)\,dX + c.c.\,.\end{aligned} \tag{35}$$

Through use of (29) this equation can be rewritten as

$$\frac{d}{dZ}\Delta\hat{X}(Z) = \frac{2}{N_0}\int \underline{\Psi}_X^{(+)}(X,Z)^*\,\hat{\Psi}^{(+)}(X,Z)\,dX + H.c. = 2\,\Delta\hat{p}(Z) \tag{36}$$

where we have used the definition (32). Note the key role played by the result (29), which is a property of any two solutions of Eqs. (24).

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